COMMON F-TEST DENOMINATOR FOR TWO-WAY INTERACTIVE BALANCED DESIGN

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Abstract

The presence of interaction in a set of data/model gives rise to the problem of difference in F-test denominator for two-way interactive balanced design. This forms the core focus of this paper. The least square estimate of the parameter and the expected mean squares for the full and reduced models were derived. When the interaction effects were removed from the full model, we obtained the reduced model and consequently introduced heterogeneity of Variance, which is a complete violation of the ANOVA assumption. An error Variance was then derived and used to normalize the effects of the distorted assumption. This was done by dividing the sum of squares of the main effects and the error of the full model by the square root of the error Variance before carrying out the test for significance on the main effects. The result shows that the reduced model yielded the same result with full model. It was recommended that the reduced model should be used whenever there is interaction in our data/model. This is true for two main reasons: - (i) It is more efficient relative to the full model. (ii) The reduced model helps to solve the problem of differences in F-test denominator.

Introduction

The main purpose of analysis of Variance is to test for statistical significance of differences in means. This is done by partitioning the total Variance into the component that is due to true random error and the components that are due to differences between means. These later Variance components are the tested for Statistical significance. If the result is significant, then we reject the null hypothesis of no significant difference between means.

In testing for statistical significance, we observed that there is no common denominator for the F-test for different statistical models. Vis-à-vis fixed, random and mixed effects models, the differences in the denominator of the F-test are as a result of the presence of interactions in the model. To enable the introduction of a common denominator, we eliminate the effects of the interaction. When the interaction was removed, a reduced model was obtained, thus introducing heterogeneity of Variance into the data, which is a complete violation of the assumption of analysis of Variance.

This paper therefore, is aimed at developing a common F-test denominator irrespective of whether the model is fixed, random or mixed effect and to derive the error variance that can be used to normalize the effect of the distorted assumption.

Full Model.

The mathematical model for the two-way interactive balanced design is given as:-

$X_{ijk} = \mu + \alpha_i + \beta_j + \lambda_{ij} + e_{ijk}$ (1)

i = 1, 2, …, p
j = 1, 2, …, q
k = 1, 2, …, r

where X_{ijk} is the kth observation in the ith level of the Factor A and the jth level of the Factor B.

μ is the general mean.

α_i is the average effect due to the ith level of the factor A.

β_j is the average effect due to the jth level of the factor B.

λ_{ij} is the interaction between the ith level of factor A and jth level of factor B.

e_{ijk} is the error associated with X_{ijk}.

From equation 1, the model is said to be a fixed effect model if the following conditions are met.

$\sum_i \alpha_i = \sum_j \beta_j = \sum_i \lambda_{ij} = \sum_j \lambda_{ij} = 0$, and $e_{ijk} \sim N(0, \sigma_e^2)$

Therefore, the expected mean squares for the main effects, interactions and error terms can be shown to be:-

$$E(MS_A) = \text{qr} \frac{qr}{p-1} \sum_{i=1}^{p} \alpha_i^2 + \sigma_e^2 .$$

$$E(MS_B) = \text{qr} \frac{pr}{q-1} \sum_{i=1}^{q} \beta_i^2 + \sigma_e^2$$

$$E(MS_\lambda) = \frac{r}{(p-1)(q-1)} \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_{ij}^2 + \sigma_e^2$$

$$E(MS_e) = \sigma_e^2$$

Similarly, the model in equation 1 is said to be a random effect model if $\alpha_i \sim N(0, \sigma_\alpha^2); \beta_j \sim N(0, \sigma_\beta^2); \lambda_{ij} \sim N(0, \sigma_\lambda^2)$ and $e_{ijk} \sim N(0, \sigma_e^2)$.

The expected mean squares for the main effects, interaction and error terms are:-

$$E(MS_A) = qr\sigma_\alpha^2 + r\sigma_\lambda^2 + \sigma_e^2$$

$$E(MS_B) = pr\sigma_\beta^2 + r\sigma_\lambda^2 + \sigma_e^2$$

$$E(MS_\lambda) = r\sigma_\beta^2 + \sigma_e^2$$

$$E(MS_e) = \sigma_e^2$$

Finally equation 1 is said to be a mixed effect model if with factor A fixed and factor B random then $\sum_i \alpha_i = \sum_i \lambda_{ij} = 0$; $\beta_j \sim N(0,\sigma_\beta^2)$ and $e_{ijk} \sim N(0,\sigma_e^2)$.

The expected mean squares for the various parameters are:-

$$E(MS_A) = \frac{qr}{p-1}\sum_{i=1}^{p}\alpha_i^2 + r\sigma_\lambda^2 + \sigma_e^2 \ .$$

$$E(MS_B) = pr\sigma_\beta^2 + \sigma_e^2$$

$$E(MS_\lambda) = r\sigma_\lambda^2 + \sigma_e^2$$

$$E(MS_e) = \sigma_e^2$$

Similarly, if factor A is random and factor B is fixed, then $\alpha i \sim N(0,\sigma_\alpha^2)$; $\sum_j \beta_j = \sum_j \lambda_{ij} = 0$; $\alpha_i \sim N(0,\sigma_\alpha^2)$; and $e_{ijk} \sim N(0,\sigma_e^2)$.

The expected mean squares are:-

$$E(MS_A) = qr\sigma_\alpha^2 + \sigma_e^2$$

$$E(MS_B) = \frac{pr}{q-1}\sum_{j=1}^{q}\beta_j^2 + r\sigma_\lambda^2 + \sigma_e^2 \ .$$

$$E(MS_\lambda) = r\sigma_\lambda^2 + \sigma_e^2$$

$$E(MS_e) = \sigma_e^2$$

The expected mean squares are summarized in a complete ANOVA table shown in table 1

Table 1Complete ANOVA table.

S.V	d.f	SS	MS	All effect fixed	All effect random	Factor A fixed & Factor B random	Factor B fixed & Factor A random
Factor A	p-1	SSα	MSα	$\frac{qr}{p-1}\sum_{i=1}^{p}\alpha_i^2 + \sigma_e^2.$	σ_e^2+rσ_λ^2 +qrσ_α^2	σ_e^2+rσ_λ^2 +	σ_e^2+rσ_λ^2 qrσ_α^2

						$\frac{qr}{p-1}\sum_{i=1}^{p}\alpha_i^2$	
Factor B	q-1	SSβ	MSβ	$\sigma_e^2 + \frac{pr}{q-1}\sum_{j=1}^{q}\beta_j^2$	σ_e^2 +r σ_λ^2 +pr σ_β^2	σ_e^2 +pr σ_β^2	σ_e^2 +r σ_λ^2 + $\frac{pr}{q-1}\sum_{j=1}^{q}\beta_j^2$
AXB Interact.	(p-1) x (q-1)	SSλ	MSλ	$\frac{r}{(p-1)(q-1)}\sum_{i=1}^{p}\sum_{j=1}^{q}\lambda_{ij}^2 + \sigma_e^2.$	σ_e^2 +r σ_λ^2	σ_e^2 +r σ_e^2	σ_e^2 +r σ_e^2
Error	pq x (r-1)	SSe	MSe	σ_e^2	σ_e^2	σ_e^2	σ_e^2
Total	pqr	SST	-	-	-	-	-

From the table above, the appropriate F-ratios for testing for the main effects for the various models are as follows:-

Model 1

H_{01} : $\alpha_1 = \alpha_2 = ... = \alpha_p$

H_{02} : $\beta_1 = \beta_2 = ... = \beta_q$

H_{03} : $\lambda_{11} = \lambda_{12} = ... = \lambda_{pq}$.

The corresponding F-ratios are:-

F-ratio for H_{01} is $\frac{MS_\alpha}{MS_e}$

F-ratio for H_{02} is $\frac{MS_\beta}{MS_e}$

F-ratio for H_{03} is $\frac{MS_\lambda}{MS_e}$

Under model 11 we have:-

H_{01} : $\sigma_{\alpha i}^2 = 0$

H_{02} : $\sigma_{\beta j}^2 = 0$

H_{03} : $\sigma_{\lambda ij}^2 = 0$ and the corresponding F-ratios are:-

F-ratio for H_{01} is $\frac{MS_\alpha}{MS_\lambda}$

F-ratio for H_{02} is $\frac{MS_\beta}{MS_\lambda}$

F-ratio for H_{03} is $\frac{MS_\lambda}{MS_e}$

Under mixed effect where factor A is fixed and factor B is random, the hypotheses and corresponding F-ratios are shown below.

H_{01} : $\alpha_1 = \alpha_2 = ... = \alpha_p$

H_{02} : $\sigma^2_{\beta j} = 0$

H_{03} : $\sigma^2_{\lambda ij} = 0$

F-ratio for H_{01} is $\frac{MS_\alpha}{MS_\lambda}$

F-ratio for H_{02} is $\frac{MS_\beta}{MS_e}$

F-ratio for H_{03} is $\frac{MS_\lambda}{MS_e}$

Similarly, when factor B is fixed and factor A is random we have:-

H_{01} : $\sigma^2_{\alpha i} = 0$

H_{02}: $\beta_1 = \beta_2 = ... = \beta_q$

H_{03} : $\sigma^2_{\lambda ij} = 0$

F-ratio for H_{01} is $\frac{MS_\alpha}{MS_e}$

F-ratio for H_{02} is $\frac{MS_\beta}{MS_\lambda}$

F-ratio for H_{03} is $\frac{MS_\lambda}{MS_e}$

From the table we can deduce that there is no clear common denominator if we wish to test for the main effects via the F-test.

The Reduced Model.

The differences in the denominator of the F-test are sequel to the presence of the interaction in the model. The situation can be remedied by eliminating the interaction effects from the model. When this is done, we have a reduced model which can be represented as :

$$Y_{ijk} = \mu + \alpha_i + \beta_j + Z_{ijk} \qquad (2)$$

i = 1, 2, …, p
j = 1, 2, …, q
k = 1, 2, …, r

The parameters have their usual meaning in Design and Analysis of experiments.

When the interaction is completely eliminated, heterogeneity of Variance is introduced into the data thereby violating the assumption of ANOVA.

In order to proceed with the Analysis of Variance, we normalize the effect of the distorted assumption of ANOVA. This is done by dividing the original data by the coefficient of square root of the error Variance given as $Var(e_{ijk}) = \sigma_e^2(1-\frac{1}{r}) = \sigma_e^2 k_{ij}$.

The original data is hence divided by the standard error given by $\frac{1}{\sqrt{k_{ij}}}$, where k_{ij} is $(1-\frac{1}{r})$.

The expected mean squares for equation 2 are shown in complete ANOVA table in table 2 below:

Table 2:Complete ANOVA table.

S.V	d.f	SS	MS	All effect fixed	All effect random	Factor A fixed & Factor B random	Factor B fixed & Factor A random
Factor A	p-1	SSα	MSα	$\frac{qr}{p-1}\sum_{i=1}^{p}\alpha_i^2 + \sigma_e^2$.	σ_e^2 +qr σ_α^2	$\frac{qr}{p-1}\sum_{i=1}^{p}\alpha_i^2$ $+\sigma_e^2$	σ_e^2+ qr σ_α^2
Factor B	q-1	SSβ	MSβ	$\sigma_e^2 +$ $\frac{pr}{q-1}\sum_{j=1}^{q}\beta_j^2$	σ_e^2 +pr σ_β^2	σ_e^2 +pr σ_β^2	$\sigma_e^2 +$ $\frac{pr}{q-1}\sum_{j=1}^{q}\beta_j^2$

AXB Interact.	(p-1) x (q-1)	SSλ	MSλ	$\frac{r}{(p-1)(q-1)} \sum_{i=1}^{p}\sum_{j=1}^{q}\lambda_{ij}^2+\sigma_e^2.$	σ_e^2 +r σ_λ^2	σ_e^2 +r σ_e^2	σ_e^2 +r σ_e^2
Error	Pq x (r-1)	SSe	MSe	σ_e^2	σ_e^2	σ_e^2	σ_e^2
Total	Pqr	SST	-	-	-	-	-

From the table above, it can be seen that the common denominator for the F-ratios for the various models is MS_e .

Efficiency of the Two Designs:
The relative efficiency of design A to another design B is defined as the ratio of the variance per unit of design A to that of design B given by:-

Efficiency (B/A) = $\frac{\text{var}(\hat{\alpha}_i)_A(f_A+1)(f_B+3)}{\text{var}(\hat{\alpha}_i)_B(f_B+1)(f_A+3)}$

Where

$Var(\hat{\alpha}_i)_A$ is the variance of $\hat{\alpha}_i$ for design A.

$Var(\hat{\alpha}_i)_B$ is the variance of $\hat{\alpha}_i$ for design B.

f_A is the error degrees of freedom for design A.

f_B is the error degrees of freedom for design B.

VARIANCE OF $\hat{t}i$

The least square estimate of $\hat{t}i$ is $\bar{X}_{i..} - \bar{X}_{...}$

$$\therefore Var(\hat{t}i) = Var(\bar{X}_{i..} - \bar{X}_{...})$$

$$= Var(\bar{X}_{i..} - \frac{\bar{X}_1 + \bar{X}_2 + ... + \bar{X}_t}{t})$$

$$= Var(\frac{t\bar{X}_{i..} - \sum_{i=1}^{t}\bar{X}_{i..}}{t})$$

$$= \frac{1}{t^2} Var\left[(t-1)\bar{X}_{i..} - \sum_{k \neq i}^{n} \bar{X}_k\right]$$

$$= \frac{1}{t^2}\left[(t-1)^2\frac{\sigma_e^2}{b} + (t-1)\frac{\sigma_e^2}{b}\right]$$

$$= \frac{\sigma_e^2}{bt^2}(t-1)[t-1+1]$$

$$= \frac{(t-1)\sigma_e^2}{tb}$$

Illustrative Example:
An engineer suspects that the surface finish of a metal part is influenced by the feed rate and the depth of cut. She selects three feed rates and randomly chooses four depths of cut. She then conducts a factorial experiment and obtains the following data.

	Depth of cut (in)				
Feed Rate (in/min)	0.15	0.18	0.20	0.25	$T_{i..}$
	74	79	82	99	
0.20	64	68	88	104	
	60	73	92	96	
	(198)	(220)	(262)	(299)	[979]
	92	98	99	104	
0.25	86	104	108	110	
	88	88	95	99	
	(266)	(290)	(302)	(313)	[1171]
	99	104	108	114	
0.30	98	99	110	111	
	102	95	99	107	
	(299)	(298)	(317)	(332)	[1246]
$T_{.j.}$	[763]	[808]	[881]	[944]	{3396}

Source: Design and analysis of experiments by D .C. Montgomery.

The model is $X_{ijk} = \mu + \alpha_i + \beta_j + \lambda_{ij} + e_{ijk}$ i = 1, 2, 3

j = 1, 2, 3, 4
k = 1, 2, 3

X_{ijk} is the kth observation in the ijth cell.

μ is the universal constant

α_i is the average effect of the feed rates.

β_j is the average effects of the depth of cut.

λ_{ij} is the interactions that exist between the feed rates and the depth of cuts.

e_{ijk} is the error associated with X_{ijk}.

$$SS_\mu = \frac{T_{...}^2}{pqr} = \frac{(3396)^2}{3x4x3} = 320356 = C$$

$$SS_\alpha = C_i - C; C_i = \frac{\sum_i T_{i..}^2}{qr}$$

$$= \frac{979^2 + 1171^2 + 1246^2}{4x3} - 320356$$

= 323516.5 – 320356 = 3160.5

$$SS_\beta = C_j - C; C_j = \frac{\sum T_{.j.}^2}{pr}$$

$$=\frac{763^2+808^2+..+944^2}{3x3}-320356$$

= 322481.11- 320356 = 2125.1

$$SS_\lambda = C_{ij}-C_i-C_j+C; C_{ij}=\frac{\sum_{ij} T_{ij}^2}{r}$$

$= \frac{198^2+220^2+...+332^2}{3}$ - 323516.5 - 322481.1 + 320356.

= 326198.7- 323516.5 - 322481.1 + 320356. = 557.1

$$SS_e = C_{ijk}-C_{ij}$$

= $74^2+64^2+...+107^2$ - 326198.7 = 689.3

The above result is summarized in the table shown below.

S.V	d.f	SS	MS	F-ratio
Feed rate (F)	2	3160.5	1580.3	$\frac{MS_\alpha}{MS_\lambda}$ = 17.01
Depth of cut (D)	3	2125.1	708.4	$\frac{MS_\beta}{MS_e}=24.9$
FxD (λ_{ij})	6	557.1	92.9	3.2
Error	24	689.3	28.7	

$$F_{2,6}^{0.05}=5.14; F_{6,24}^{0.05}=2.51.$$

Both the main effects of the feed rate and depth of cut and the interaction are significant.

To remove the interaction so as to have a common denominator for the F-test, we shall divide the original data by $\sqrt{k_{ij}}$.