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The Alpha Power Topp-Leone Distribution: Properties, Simulations and Applications

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Abstract

This paper presents an extended lifetime probability distribution based on the alpha power transformation. We refer to the proposed distribution as "the Alpha Power Topp-Leone (APTL) distribution". Mathematical properties of the APTL distribution such as the density and cumulative distribution functions, survival and hazard rate functions, quantile function, median, moments and its relative measures, probability weighted moment, moment generating function, Renyi entropy, and the distribution of order statistics were derived. The method of maximum likelihood estimation was employed to estimate the unknown parameters of the APTL distribution. Finally, we used two real data sets obtained from the literature to illustrate the applicability of the APTL distribution in real-life data fitting.

Keywords

Alpha Power Transformation, Topp-Leone Distribution, Quantiles, Moments

1. Introduction

Lifetime distributions play important roles in the statistical modelling of real-life phenomena such as survival studies in biological sciences and reliability theory in engineering. Many lifetime distributions have been developed and widely applied to model real data sets in the field of biological sciences, engineering, actuarial sciences, demography, and more. In many cases, there have been scenarios where the classical lifetime distributions fail to provide a good fit in data analysis. In other to address this pitfall, the attention of researchers has recently been focused on the need to develop more flexible distributions that can handle any sophisticated data. Several methods of generalization of lifetime distributions have been introduced in the literature. Some of these methods include the exponentiated Weibull family by [1], the Marshall-Olkin extended family by [2], the transmuted-G family by [3], the Kumaraswamy-G family by [4], the beta-G family by [5], the T-X family by [6], the Weibull-G family by [7], the T-R{Y} family by [8] and many others.

Recently, [9] introduced a new method for generating lifetime distributions and called it the "alpha-power transformation method".

Let $F(x)$ be the Cumulative Distribution Function (CDF) of a continuous random variable *X*, then the alpha-power transformation of $F(x)$ for $x \in R$ is defined as

$$F_{APT}(x,\alpha,\xi) = \begin{cases} \dfrac{\alpha^{F(x,\xi)}-1}{\alpha-1}, & \text{if } \alpha>0,\ \alpha\neq 1 \\ F(x,\xi), & \text{if } \alpha=1 \end{cases}, \tag{1}$$

and the corresponding Probability Density Function (PDF) associated with (1) is defined as

$$f_{APT}(x,\alpha,\xi) = \begin{cases} \dfrac{\log\alpha}{\alpha-1} f(x,\xi)\alpha^{F(x,\xi)}, & \text{if } \alpha>0,\ \alpha\neq 1 \\ f(x,\xi), & \text{if } \alpha=1 \end{cases}, \tag{2}$$

where $F(x,\xi)$ is known as the baseline distribution with parameter vector ξ. The authors considered the CDF of the exponential distribution as the baseline distribution in (1) and (2) to develop the Alpha-Power Exponential (APE) distribution.

The alpha-power transformation method defined in (1) and (2) has attracted the attention of researchers to introduce more flexible generalizations of existing classical lifetime distributions. [10] proposed the alpha-power Raleigh distribution, [11] introduced the alpha-power Weibull distribution, [12] developed the alpha-power Lindley distribution, [13] introduced the alpha-power transformed extended exponential distribution, [14] proposed the alpha-power inverted exponential distribution, [15] studied the alpha-power inverse Lindley distribution, [16] developed the alpha-power transformed power Lindley distribution, [17] proposed the alpha-power Pareto distribution, [18] developed the alpha-power inverse Weibull distribution, [19] proposed the alpha-power inverse Lomax distribution, [20] introduced the alpha-power Gompertz distribution, amongst many others.

In this paper, we employed the same method of generalization and in particular, considered the case where the baseline distribution $F(x,\xi)$ follows the one-parameter Topp-Leone distribution.

The one-parameter Topp-Leone distribution with shape parameter $\lambda>0$ is defined by its cumulative distribution function as

$$F(x) = \left[1-(1-x)^2\right]^{\lambda}, \tag{3}$$

and the density function obtained as

$$f(x) = 2\lambda(1-x)\left[1-(1-x)^2\right]^{\lambda-1}. \tag{4}$$

By inserting (3) and (4) into (1) and (2), we define the cumulative distribution function of the Alpha Power Topp-Leone (APTL) distribution as

$$F_{APTL}(x,\alpha,\lambda)=\begin{cases}\dfrac{\alpha^{\left[1-(1-x)^2\right]^{\lambda}}-1}{\alpha-1}, & \text{if } \alpha>0,\ \alpha\neq 1\\ \left[1-(1-x)^2\right]^{\lambda}, & \text{if } \alpha=1\end{cases}. \tag{5}$$

The corresponding density function of the APTL distribution is defined as

$$f_{APTL}(x,\alpha,\lambda)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}2\lambda(1-x)\left[1-(1-x)^2\right]^{\lambda-1}\alpha^{\left[1-(1-x)^2\right]^{\lambda}}, & \text{if } \alpha>0,\ \alpha\neq 1\\ 2\lambda(1-x)\left[1-(1-x)^2\right]^{\lambda-1}, & \text{if } \alpha=1\end{cases}. \tag{6}$$

The density function in (6) can be expressed in series representation following the generalized binomial expansion defined as

$$(1-Z)^{\alpha-1}=\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j Z^j. \tag{7}$$

Using the Taylor's series expansion for the expression $\alpha^{\left[1-(1-x)^2\right]^{\lambda}}$, we have

$$\alpha^{\left[1-(1-x)^2\right]^{\lambda}}=\sum_{j=0}^{\infty}\frac{\left[\log(\alpha)\right]^j\left[1-(1-x)^2\right]^{\lambda j}}{j!},$$

from (7),

$$\left[1-(1-x)^2\right]^{\lambda(j+1)-1}=\sum_{k=0}^{\infty}\binom{\lambda(j+1)-1}{k}(-1)^k(1-x)^{2k},$$

substituting these expressions into (6), we have

$$f_{APTL}(x)=\frac{2\lambda}{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\lambda(j+1)-1}{k}\frac{(-1)^k\left[\log(\alpha)\right]^{j+1}}{j!}(1-x)^{2k+1}. \tag{8}$$

We noticed at the time of writing, that the alpha power transformation method has not been employed to generalize any unit distribution, thus, the motivation for this paper. It is, therefore, important to remark that the APTL distribution is the first-lifetime distribution belonging to the alpha power transformed family of distributions that has its support on a unit interval [0, 1]. Other non-nested generalized lifetime distributions with support [0, 1] are found in the works of [21]-[27]. It is hoped that the APTL distribution will be a strong competitor unit distribution in fitting data sets defined on a unit interval.

The organization of this paper is structured as follows: In Section 2, we present the mathematical properties of the proposed APTL distribution. Section 3 discusses the maximum likelihood method of estimation of the unknown parameters of the proposed APTL distribution. In Section 4, we considered two data sets defined on a unit interval to illustrate the applicability of the proposed APTL distribution in real-life data fitting. Finally, in Section 5, we gave a concluding remark.

2. Mathematical Properties of the APTL Distribution

In this Section, we studied some mathematical properties of the APTL distribution which include; the survival, hazard rate and quantile functions, moments, moment generating function, probability weighted moment, Renyi entropy, and the distribution of order statistics.

2.1. Survival, Hazard Rate and Quantile Functions of the APTL Distribution

The survival, hazard rate and quantile functions of the APTL distribution are respectively defined from (3) and (4) as follows

$$S_{APTL}(x,\alpha,\lambda)=1-F(x,\alpha,\lambda)$$
$$=\begin{cases}\dfrac{\alpha\left(1-\alpha^{\left[1-(1-x)^2\right]^{\lambda}-1}\right)}{\alpha-1}, & \text{if } \alpha>0,\ \alpha\neq 1\text{'}\\ 1-\left[1-(1-x)^2\right]^{\lambda}, & \text{if } \alpha=1\end{cases} \tag{9}$$

$$h_{APTL}(x,\alpha,\lambda)=\frac{f_{APTL}(x,\alpha,\lambda)}{S_{APTL}(x,\alpha,\lambda)}$$
$$=\begin{cases}\dfrac{\log(\alpha)2\lambda(1-x)\left[1-(1-x)^2\right]^{\lambda-1}\alpha^{\left[1-(1-x)^2\right]^{\lambda}-1}}{\left(1-\alpha^{\left[1-(1-x)^2\right]^{\lambda}-1}\right)}, & \text{if } \alpha>0,\ \alpha\neq 1\\ \dfrac{2\lambda(1-x)\left[1-(1-x)^2\right]^{\lambda-1}}{1-\left[1-(1-x)^2\right]^{\lambda}}, & \text{if } \alpha=1\end{cases} .\tag{10}$$

The quantile function of the APTL distribution is obtained by solving the system of equation $F(x,\alpha,\lambda)=q$, $0<q<1$, *i.e.*

$$\frac{\alpha^{\left[1-(1-x)^2\right]^{\lambda}}-1}{\alpha-1}=q,$$

$$\left[1-(1-x)^2\right]^{\lambda}\log(\alpha)=\log\left[1+q(\alpha-1)\right],$$

$$(1-x)^2=1-\left[\frac{\log\left[1+q(\alpha-1)\right]}{\log(\alpha)}\right]^{1/\lambda},$$

$$x_q=1-\left[1-\left[\frac{\log\left[1+q(\alpha-1)\right]}{\log(\alpha)}\right]^{1/\lambda}\right]^{1/2}. \tag{11}$$

The median of the APTL distribution is obtained by substituting $q=0.5$ in as

$$x_{0.5}=1-\left[1-\left[\frac{\log(\alpha+1)-\log(2)}{\log(\alpha)}\right]^{1/\lambda}\right]^{1/2}.$$

Table 1 presents numerical computation of some quantiles from the APTL distribution for varying values of the parameters.

Table 1 validates the claim that random samples from the APTL distribution fall within the unit interval.

Figure 1 and **Figure 2**, respectively, display some graphical presentation of the density and hazard rate functions of the APTL distribution for varying values of the parameters.

Figure 1 shows that the density plot of the APTL distribution accommodates a decreasing, left-skewed, right-skewed and symmetric shapes, whereas, the plots displayed in **Figure 2** indicates that the hazard function of the APTL distribution exhibits an increasing, bathtub and upside-down bathtub shaped hazard properties.

2.2. The r^{th} Moment of the APTL Distribution

Let X be a continuous random variable following a known probability distribution with density function $f(x)$, then the r^{th} moment about the origin of X is defined as

Table 1. Some quantiles of the APTL distribution for varying values of the parameters.

U	$(\alpha=0.5,\lambda=4)$	$(\alpha=0.5,\lambda=4)$	$(\alpha=0.5,\lambda=4)$	$(\alpha=0.5,\lambda=4)$
0.05	0.1006	0.2498	0.1429	0.3036
0.1	0.1468	0.3083	0.2068	0.3747
0.2	0.2189	0.3871	0.3021	0.4672
0.3	0.2818	0.4485	0.3797	0.5356
0.4	0.3423	0.5033	0.4493	0.5934
0.5	0.4035	0.5557	0.5151	0.6458
0.6	0.4683	0.6088	0.5798	0.6958
0.7	0.5399	0.6652	0.6464	0.7458
0.8	0.6238	0.7290	0.7187	0.7990
0.9	0.7330	0.8095	0.8058	0.8620

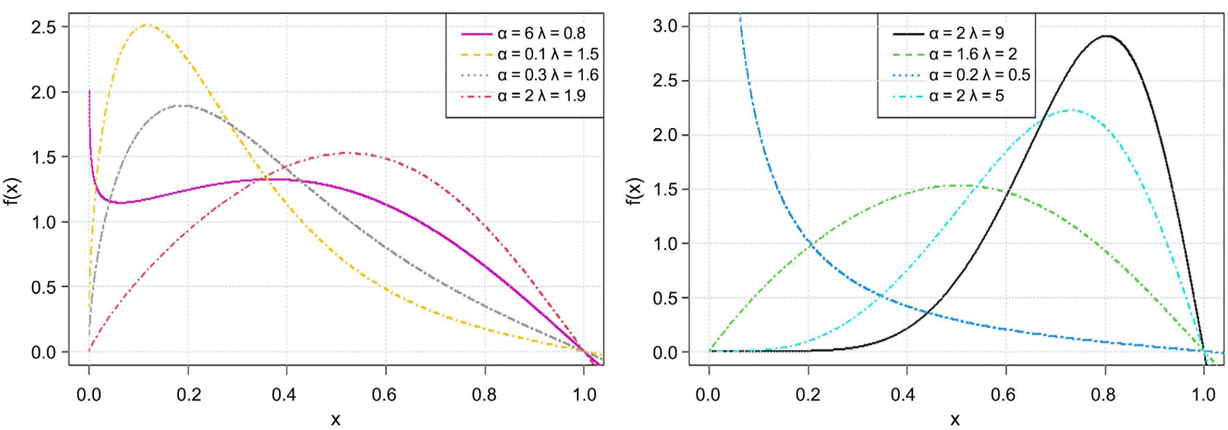

Figure 1. Density plots of the APTL distribution.

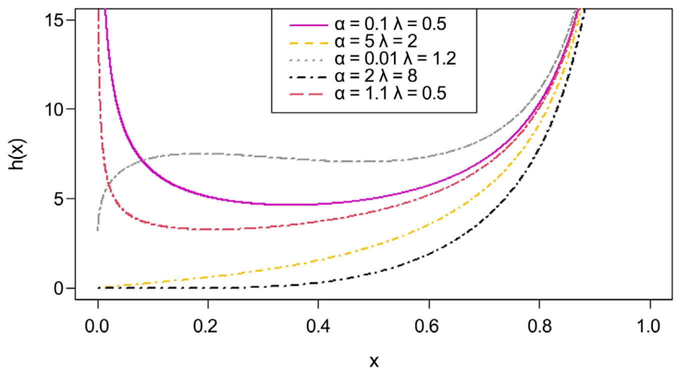

Figure 2. Hazard plots of the APTL distribution.

$$E\left(X^r\right)=\int_{-\infty}^{\infty} x^r f\left(x\right)\mathrm{d}x, \quad r=1,2,3,4,\cdots. \tag{10}$$

By inserting the density function in (8) into (10), we obtain an expression for the r^{th} moment about the origin of the APTL distribution as

$$E\left(X^r\right)=\frac{2\lambda}{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\lambda(j+1)-1}{k}\frac{(-1)^k\left[\log(\alpha)\right]^{j+1}}{j!}\int_0^1 x^r\left(1-x\right)^{2k+1}\mathrm{d}x, \tag{11}$$

using the generalized binomial expansion in (7), we obtain

$$\left(1-x\right)^{2k+1}=\sum_{m=0}^{\infty}\binom{2k+1}{m}(-1)^m x^m,$$

so that (11) now becomes,

$$\begin{aligned} E\left(X^r\right)&=\frac{2\lambda}{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{m}\frac{(-1)^{k+m}\left[\log(\alpha)\right]^{j+1}}{j!}\int_0^1 x^{r+m}\mathrm{d}x,\\ &=\frac{2\lambda}{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{m}\frac{(-1)^{k+m}\left[\log(\alpha)\right]^{j+1}}{j!(r+m+1)}. \end{aligned} \tag{12}$$

Since,

$$\int_0^1 x^{r+m}\mathrm{d}x=\frac{1}{r+m+1}.$$

Consequently, the first four r^{th} moment about the origin of the APTL distribution are obtained from (12) as

$$\mu_1'=\mu=\frac{2\lambda}{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{m}\frac{(-1)^{k+m}\left[\log(\alpha)\right]^{j+1}}{j!(2+m)},$$

$$\mu_2'=\frac{2\lambda}{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{m}\frac{(-1)^{k+m}\left[\log(\alpha)\right]^{j+1}}{j!(3+m)},$$

$$\mu_3'=\frac{2\lambda}{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{m}\frac{(-1)^{k+m}\left[\log(\alpha)\right]^{j+1}}{j!(4+m)},$$

$$\mu_4'=\frac{2\lambda}{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{m}\frac{(-1)^{k+m}\left[\log(\alpha)\right]^{j+1}}{j!(5+m)}.$$

Other moment related measures such as the variance (σ^2), skewness (S_k) and kurtosis (K_s) are obtained using the following expressions

$$\text{variance}\left(\sigma^2\right) = \mu_2' - \left(\mu_1'\right)^2,$$

$$\text{skewness}\left(S_k\right) = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\left(\mu_1'\right)^3}{\left(\mu_2' - \left(\mu_1'\right)^2\right)^{\frac{3}{2}}},$$

$$\text{kurtosis}\left(K_s\right) = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_3'\left(\mu_1'\right)^2 - 3\left(\mu_1'\right)^4}{\left(\mu_2' - \left(\mu_1'\right)^2\right)^2}.$$

Table 2 shows the numerical computation of the first four r^{th} moments, variance (σ^2), measures of skewness (S_k) and kurtosis(K_s) of the APTL distribution.

Table 2 reveals that the APTL distribution can be negatively skewed $\left(S_k < 0\right)$, positively skewed $\left(S_k > 0\right)$, approximately symmetric $\left(S_k \approx 0\right)$, leptokurtic $\left(K_s > 3\right)$, platykurtic $\left(K_s < 3\right)$ and mesokurtic $\left(K_s \approx 3\right)$. This result supports the claim in Figure 1.

2.3. Moment Generating Function of the APTL Distribution

Generating functions are known to determine the distribution of a random variable, while the moments of a random variable can be obtained from either the derivatives of the generating function, or, the coefficients in the power series expansion of the generating function [28] [29].

Let X be a continuous random variable following a known probability distribution with density function $f\left(x\right)$, then the moment generating function of X is defined as

$$M_X\left(t\right) = E\left[\mathrm{e}^{tx}\right] = \int_{-\infty}^{\infty} \mathrm{e}^{tx} f\left(x\right)\mathrm{d}x. \tag{13}$$

The definition in (13) was extended by [30] through a generalized method for generating moments of continuous random variables, including positive and negative real number powers of the random variable.

Table 2. Moments of the APTL distribution at varying values of the parameters.

α	λ	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	S_k	K_s
0.2	0.5	0.1301	0.0487	0.0256	0.0159	0.0318	1.9392	6.5900
	1.0	0.2341	0.0968	0.0521	0.0324	0.0420	1.1358	3.6468
	3.0	0.4526	0.2425	0.1462	0.0960	0.0377	0.3225	2.4520
0.9	0.5	0.2084	0.0922	0.0524	0.0339	0.0488	1.1928	3.6056
	1.0	0.3263	0.1614	0.0963	0.0639	0.0549	0.6055	2.4246
	3.0	0.5367	0.3291	0.2204	0.1570	0.0411	−0.0353	2.1961
1.5	0.5	0.2391	0.1107	0.0643	0.0422	0.0535	0.9886	3.0990
	1.0	0.3606	0.1874	0.1150	0.0777	0.0574	0.4399	2.2178
	3.0	0.5663	0.3613	0.2491	0.1814	0.0406	−0.1825	2.3087

By the Maclaurin series expansion of the exponential function, we have

$$e^{tx}=\sum_{n=o}^{\infty}\frac{(tx)^n}{n!},$$

so that (13) now becomes

$$M_X(t)=\sum_{n=o}^{\infty}\frac{t^n}{n!}\int_{-\infty}^{\infty}x^n f(x)\mathrm{d}x. \tag{14}$$

Hence, by inserting the density function in (8) into (14), we obtain the moment generating function of the APTL distribution as

$$\begin{aligned}M_X(t)&=\frac{2\lambda}{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{\lambda(j+1)-1}{k}\frac{(-1)^k[\log(\alpha)]^{j+1}}{j!n!}\int_0^1 x^n(1-x)^{2k+1}\mathrm{d}x,\\&=\frac{2\lambda}{\alpha-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\binom{\lambda(j+1)-1}{k}\binom{2k+1}{p}\frac{(-1)^{k+p}[\log(\alpha)]^{j+1}t^n}{j!n!(n+p+1)}.\end{aligned} \tag{15}$$

2.4. Probability Weighted Moments (PWMs) of the APTL Distribution

Suppose X is a continuous random variable from a known probability distribution with density function $f(x)$, and cumulative distribution function $F(x)$, [31] defined the $(s,r)^{th}$ PWMs of X as

$$\rho_{s,r}=E\left(X^rF(x)^s\right)=\int_{-\infty}^{\infty}x^r f(x)F(x)^s\,\mathrm{d}x, \tag{16}$$

combining the expression in (5) and (6), we have

$$f(x)F(x)^s=\frac{\log(\alpha)2\lambda}{(\alpha-1)^{s+1}}(1-x)\left[1-(1-x)^2\right]^{\lambda-1}\alpha^{\left[1-(1-x)^2\right]^{\lambda}}\left[\alpha^{\left[1-(1-x)^2\right]^{\lambda}}-1\right]^s. \tag{17}$$

Using the generalized binomial expansion on the term $\left[\alpha^{\left[1-(1-x)^2\right]^{\lambda}}-1\right]^s$, we obtain

$$\left[\alpha^{\left[1-(1-x)^2\right]^{\lambda}}-1\right]^s=\sum_{j=0}^{\infty}\binom{s}{j}(-1)^j\alpha^{j\left[1-(1-x)^2\right]^{\lambda}},$$

$$\alpha^{(j+1)\left[1-(1-x)^2\right]^{\lambda}}=\sum_{k=0}^{\infty}\frac{(j+1)^k[\log(\alpha)]^k\left[1-(1-x)^2\right]^{\lambda k}}{k!},$$

$$\left[1-(1-x)^2\right]^{\lambda(k+1)-1}=\sum_{m=0}^{\infty}\binom{\lambda(k+1)-1}{m}(-1)^m(1-x)^{2m}.$$

So that (17) now becomes,

$$\begin{aligned}f(x)F(x)^s=&\frac{2\lambda}{(\alpha-1)^{s+1}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{s}{j}\binom{\lambda(k+1)-1}{m}\\&\times\frac{(-1)^{m+j}(j+1)^k[\log(\alpha)]^{k+1}}{k!}(1-x)^{2m+1},\end{aligned} \tag{18}$$

substitute (18) into (16) and employing similar approach used in the moment, we obtain the $(s,r)^{th}$ PWMs of the APTL distribution as

$$\rho_{s,r} = \frac{2\lambda}{(\alpha-1)^{s+1}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \binom{s}{j}\binom{\lambda(k+1)-1}{m}\binom{2m+1}{n} \frac{(-1)^{m+j+n}(j+1)^k\left[\log(\alpha)\right]^{k+1}}{k!(r+n+1)}.$$

2.5. Renyi Entropy of the APTL Distribution

Entropy is an imperative concept in probability theory with extensive applications in various areas such as physics, communication, signal processing, etc. An entropy of a random variable X is defined as the degree of uncertainty associated with X. The Renyi entropy of X is defined in [32] as

$$\tau_R(\xi) = \frac{1}{1-\xi}\log\int_{-\infty}^{\infty} f^{\xi}(x)\mathrm{d}x, \quad \xi > 0,\ \xi \neq 1. \tag{19}$$

Suppose X is associated with the density function defined in (6), then the Renyi entropy of X is obtained as follows

$$f^{\xi}(x) = \left(\frac{\log(\alpha)2\lambda}{\alpha-1}\right)^{\xi}(1-x)^{\xi}\left[1-(1-x)^2\right]^{\xi(\lambda-1)}\alpha^{\xi\left[1-(1-x)^2\right]^{\lambda}},$$

$$\alpha^{\xi\left[1-(1-x)^2\right]^{\lambda}} = \sum_{j=0}^{\infty}\frac{\left(\xi\log(\alpha)\right)^j}{j!}\left[1-(1-x)^2\right]^{\lambda j},$$

$$\left[1-(1-x)^2\right]^{\lambda(\xi+j)-\xi} = \sum_{k=0}^{\infty}\binom{\lambda(\xi+j)-\xi}{k}(-1)^k(1-x)^{2k},$$

$$(1-x)^{2k+\xi} = \sum_{m=0}^{\infty}\binom{2k+\xi}{m}(-1)^m x^m,$$

substituting these expressions into (19), yields

$$\tau_R(\xi) = \frac{1}{1-\xi}\log\left[\left(\frac{2\lambda}{\alpha-1}\right)^{\xi}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{\lambda(\xi+j)-\xi}{k}\binom{2k+\xi}{m}\frac{(-1)^{k+m}\xi^j\left[\log(\alpha)\right]^{\xi+j}}{j!(m+1)}\right]. \tag{20}$$

2.6. Distribution of Order Statistics of the APTL Distribution

Let $(X_1, X_2, \cdots, X_n)$ be random samples of size n from a known probability distribution. Suppose $X_{r:n}$ denotes the r^{th} order statistics, then the density function of $X_{r:n}$ is defined by

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j f(x)F(x)^{r+j-1}. \tag{21}$$

Using similar approach in PWMs, we define the distribution of order statistics of APTL distribution as follows

$$f(x)F(x)^{r+j-1} = \frac{\log(\alpha)2\lambda}{(\alpha-1)^{r+j}}(1-x)\left[1-(1-x)^2\right]^{\lambda-1}\alpha^{\left[1-(1-x)^2\right]^{\lambda}} \times\left[\alpha^{\left[1-(1-x)^2\right]^{\lambda}}-1\right]^{r+j-1},$$

$$\left[\alpha^{\left[1-(1-x)^2\right]^\lambda}-1\right]^{r+j-1}=\sum_{k=0}^{r+j-1}\binom{r+j-1}{k}(-1)^k\alpha^{k\left[1-(1-x)^2\right]^\lambda},$$

$$\alpha^{(k+1)\left[1-(1-x)^2\right]^\lambda}=\sum_{m=0}^{\infty}\frac{(k+1)^m\left[1-(1-x)^2\right]^{\lambda m}\left[\log(\alpha)\right]^m}{m!},$$

$$\left[1-(1-x)^2\right]^{\lambda(m+1)-1}=\sum_{n=0}^{\lambda(m+1)-1}\binom{\lambda(m+1)-1}{n}(-1)^n(1-x)^{2n}.$$

Inserting these expressions into (21), yields

$$f_{r:n}(x)=\frac{1}{B(r,n-r+1)}\sum_{j=0}^{n-r}\sum_{k=0}^{r+j-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\lambda(m+1)-1}\binom{n-r}{j}\binom{r+j-1}{k}\binom{\lambda(m+1)-1}{n}\times\frac{(-1)^{j+k+n}(k+1)^m\left[\log(\alpha)\right]^{m+1}}{m!(\alpha-1)^{r+j}}(1-x)^{2n+1}. \tag{22}$$

The s^{th} moment of the r^{th} order statistics of $X_{r:n}$ is obtained as

$$E\left(X_{r:n}^s\right)=\frac{2\lambda}{B(r,n-r+1)}\sum_{j=0}^{n-r}\sum_{k=0}^{r+j-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\lambda(m+1)-1}\sum_{p=0}^{2n+1}\binom{n-r}{j}\binom{r+j-1}{k}\binom{\lambda(m+1)-1}{n}\times\binom{2n+1}{p}\frac{(-1)^{j+k+n+p}(k+1)^m\left[\log(\alpha)\right]^{m+1}}{m!(\alpha-1)^{r+j}(s+p+1)}. \tag{23}$$

3. Parameter Estimation

Maximum Likelihood Estimation

Let $(x_1,x_2,\cdots,x_n)$ be a random sample of size n from the APTL distribution with density function $f(x)$, defined in (6), then the likelihood function is obtained as

$$L(x)=\prod_{i=1}^{n}f(x_i),\\ =\left(\frac{\log(\alpha)}{\alpha-1}\right)^n(2\lambda)^n\prod_{i=1}^{n}(1-x_i)\prod_{i=1}^{n}\left(\left[1-(1-x_i)^2\right]^{\lambda-1}\right)\prod_{i=1}^{n}\left(\alpha^{\left[1-(1-x_i)^2\right]^\lambda}\right), \tag{24}$$

taking the natural logarithm of (24), we obtain

$$\ell(x)=n\log\left[\log[\alpha]\right]-n\log[\alpha-1]+n\log[2\lambda]+\sum_{i=1}^{n}\log[1-x_i]\\ +(\lambda-1)\sum_{i=1}^{n}\log\left[1-(1-x_i)^2\right]+\log[\alpha]\sum_{i=1}^{n}\left(1-(1-x_i)^2\right)^\lambda, \tag{25}$$

minimizing the log-likelihood function in (25) with respect to the parameters, yields

$$\frac{\partial\ell(x)}{\partial\alpha}=\frac{n}{\alpha\log[\alpha]}-\frac{n}{\alpha-1}+\frac{1}{\alpha}\sum_{i=1}^{n}\left(1-(1-x_i)^2\right)^\lambda,$$

$$\frac{\partial\ell(x)}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\log\left[1-(1-x_i)^2\right]\\ +\log[\alpha]\sum_{i=1}^{n}\log\left[1-(1-x_i)^2\right]\left(1-(1-x_i)^2\right)^\lambda.$$

This system of equations does not exist in closed form, and thus, cannot be solved analytically. In such case, an iterative scheme is adopted. Here, the "***fit-distrplus***" package in R software program is employed to obtain the solutions of the system of equations.

4. Data Analysis

In this section, we considered two real data sets defined on unit interval to illustrate the applicability of the APTL distribution in real-life data fitting. The fit of the APTL distribution will be compared with ones attained by some existing unit distributions. Moe specifically, the competitor distributions are defined in terms of their density function as

1) Unit-Weibull Distribution (UWD) proposed by [26],

$$f(x,\alpha,\beta)=\frac{1}{x}\alpha\beta(-\log x)^{\beta-1}\exp\left[-\alpha(-\log x)^{\beta}\right];$$

2) Unit-Gompertz Distribution (UGD) proposed by [33],

$$f(x,\alpha,\beta)=\alpha\beta x^{-(\alpha+1)}\mathrm{e}^{-\beta\left(x^{-\alpha}-1\right)};$$

3) Log-weighted exponential distribution proposed by [34],

$$f(x,\alpha,\beta)=\frac{\alpha+1}{\alpha}\beta\exp(-\beta x)\left(1-\mathrm{e}^{-\alpha\beta x}\right);$$

4) Topp-Leone Distribution (TLD) proposed by [35],

$$f(x,\lambda)=2\lambda(1-x)\left[1-(1-x)^{2}\right]^{\lambda-1}.$$

Data Set 1:

The first data set comprises of trade share data reported in [23]. The trade share data set consists of the following values:

0.140501976, 0.156622976, 0.157703221, 0.160405084, 0.160815045, 0.22145839, 0.299405932, 0.31307286, 0.324612707, 0.324745566, 0.329479247, 0.330021679, 0.337879002, 0.339706242, 0.352317631, 0.358856708, 0.393250912, 0.41760394, 0.425837249, 0.43557933, 0.442142904, 0.444374621, 0.450546652, 0.4557693, 0.46834656, 0.473254889, 0.484600782, 0.488949597, 0.509590268, 0.517664552, 0.527773321, 0.534684658, 0.543337107, 0.544243515, 0.550812602, 0.552722335, 0.56064254, 0.56074965, 0.567130983, 0.575274825, 0.582814276, 0.603035331, 0.605031252, 0.613616884, 0.626079738, 0.639484167, 0.646913528, 0.651203632, 0.681555152, 0.699432909, 0.704819918, 0.729232311, 0.742971599, 0.745497823, 0.779847085, 0.798375845, 0.814710021, 0.822956383, 0.830238342, 0.834204197, 0.979355395. Details of this data set can be accessed in [36].

Data Set 2:

The second data set contains records of ordered failure of components used in [37]. The data sets are as follows:

0.0009, 0.004, 0.0142, 0.0221, 0.0261, 0.0418, 0.0473, 0.0834, 0.1091, 0.1252, 0.1404, 0.1498, 0.175, 0.2031, 0.2099, 0.2168, 0.2918, 0.3465, 0.4035, 0.6143.

Figure 3 presents the box plots for the two data sets.

Figure 3 reveals that Data Sets 1 and 2 are both right-skewed. A close look at the box plot for Data Set 1 suggests the presence of an outlier, while the box plot for Data Set 2 indicates that there are no outliers in the data set.

The Log-likelihood (LogL), Akaike Information Criterion (AIC), Kolmogorov-Smirnov (K-S), Crammer-von-Mises (W^*) and Anderson Darling (A^*) test statistics with their corresponding p-value will be considered as model selection criteria.

In model selection based on the aforementioned criteria, the model with the highest value of log-likelihood and the least value in terms of the Akaike Information Criterion (AIC), Kolmogorov-Smirnov (K-S), Crammer-von-Mises (W^*) and Anderson Darling (A^*) test statistics is considered to be the most appropriate model to fit the data set under study. Table 3 and Table 4 indicate that the proposed APTL distribution has the highest log-likelihood value as well as the least value in terms of the Akaike Information Criterion (AIC), Kolmogorov-Smirnov (K-S), Crammer-von-Mises (W^*) and Anderson Darling (A^*) test statistics values, thus, making the proposed APTL distribution more appropriate model than the competitor distributions in fitting the two real-life data sets. Figure 4 and Figure 5 display the empirical and fitted PDFs and CDFs of the models for the two data sets.

Table 3. Summary statistics for Data Set 1.

Model	*Estimates*	*LogL*	*AIC*	*K-S*	W^*	A^*
APTLD	$\alpha = 0.0958$ $\lambda = 0.8349$	16.8009	−29.6017 (0.9237)	0.1157 (0.9464)	0.0381 (0.9817)	0.2259
UWD	$\alpha = 0.1598$ $\beta = 1.7269$	16.4575	−28.9150 (0.8335)	0.1318 (0.8625)	0.0531 (0.9284)	0.3117
UGD	$\alpha = 0.7741$ $\beta = 0.2782$	14.7625	−25.5251 (0.7093)	0.1494 (0.5911)	0.0996 (0.5991)	0.6509
LWED	$\alpha = 0.0003$ $\lambda = 0.7807$	16.4330	−28.8659 (0.8120)	0.1351 (0.8689)	0.0521 (0.9069)	0.3371
TLD	$\lambda = 0.5112$	15.6167	−29.2337 (0.4481)	0.1848 (0.5390)	0.1114 (0.5878)	0.6638

Table 4. Summary statistics for Data Set 2.

Model	*Estimates*	*LogL*	*AIC*	*K-S*	W^*	A^*
APTLD	$\alpha = 0.4159$ $\lambda = 3.3735$	14.4209	−24.8417 (0.9808)	0.0575 (0.9262)	0.0416 (0.8713)	0.3764
UWD	$\alpha = 1.3396$ $\beta = 1.7346$	14.2436	−24.4871 (0.9210)	0.0682 (0.8049)	0.0617 (0.7427)	0.5034
UGD	$\alpha = 0.6162$ $\beta = 1.0921$	10.8759	−17.7518 (0.4235)	0.1098 (0.2535)	0.2076 (0.1897)	1.4468
LWED	$\alpha = -0.00002$ $\lambda = 2.6578$	13.0830	−22.1659 (0.5108)	0.1025 (0.4376)	0.1356 (0.4408)	0.8576
TLD	$\lambda = 2.7391$	13.9202	−24.8404 (0.7258)	0.0859 (0.6495)	0.0878 (0.6538)	0.5931

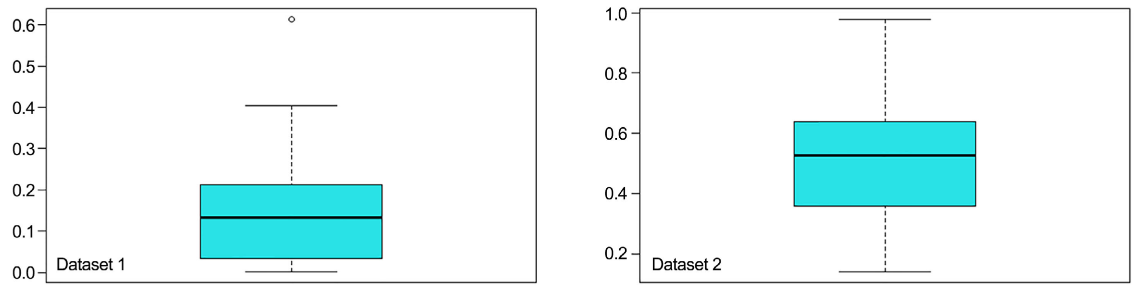

Figure 3. Box plots for the two data sets.

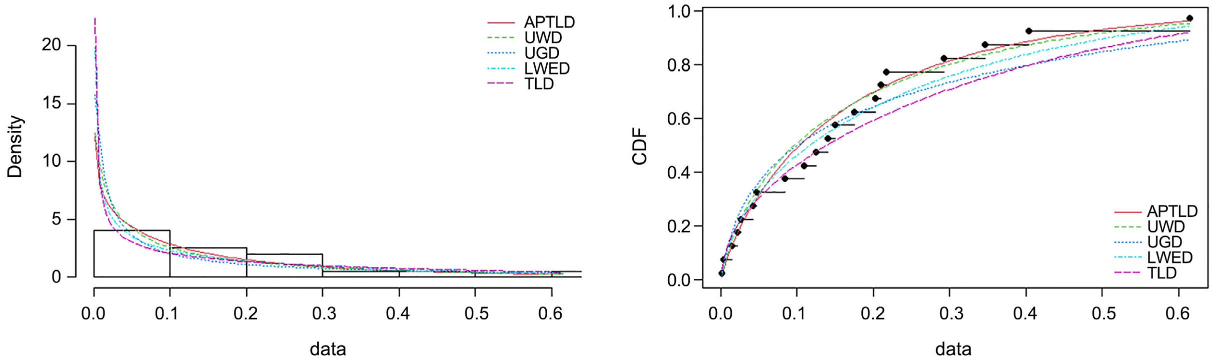

Figure 4. The empirical and fitted PDFs and CDFs of the models for Data Set 1.

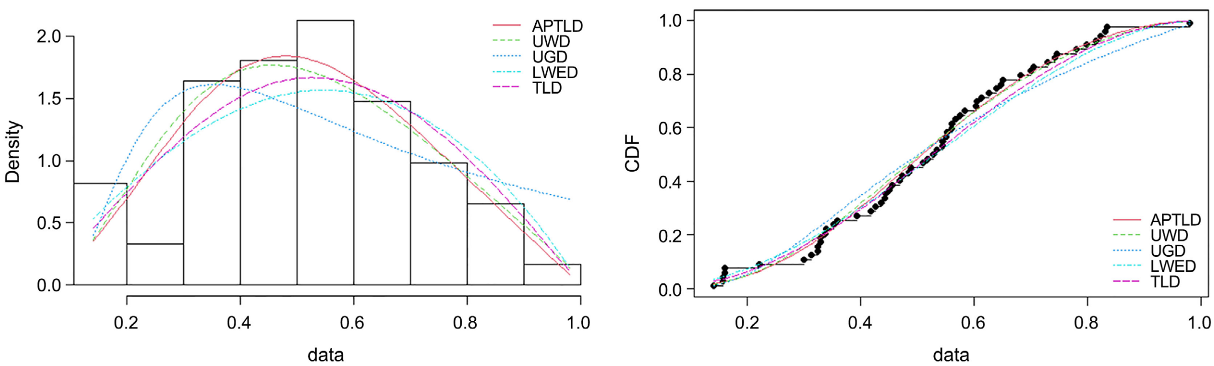

Figure 5. The empirical and fitted PDFs and CDFs of the models for Data Set 2.

In Figure 4 and Figure 5, we observe that the fit of the APTL distribution matches closer to the fit of the data sets than the rest competitor distributions. This result further supports the claim that the APTL distribution provides the best fit for the two data sets under study.

5. Conclusion

In this paper, we have introduced a new probability distribution which we called "the Alpha Power Topp-Leone (APTL) distribution". Some mathematical properties of the APTL distribution were derived. The graphical plots of the density function indicate that the APTL distribution exhibits a decreasing (reversed-J), left-skewed, right-skewed unimodal, and symmetric shapes, while the hazard rate function displays an increasing, bathtub, and inverted bathtub (upside-down) shapes. These features make the APTL distribution a suitable model for fitting datasets that exhibits these traits. We employed the method of maximum likelihood estimation to estimate the unknown parameters of the APTL distribution. Finally, two real data sets were used to illustrate the potentiality of the APTL distribution in real-life data fitting defined on a unit interval.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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