A REVIEW ON THE LINDLEY FAMILY OF DISTRIBUTIONS

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Abstract
The Lindley distribution is one of the statistical distributions widely used in modeling real lifetime data, but notably due to the monotonic property of the hazard rate function, the distribution fails to provide good fit for certain data set of interest. In order to address this situation, researchers have keenly been attracted to the act of developing generalized distributions with the aim of increasing the flexibility of the classical Lindley distribution. This paper presents a review on some methods of developing new generalization of lifetime distributions, in particular, the Lindley family of distributions. The applicability of some well-known generalized Lindley distributions is illustrated using a real data set, and the Maximized log-likelihood (Log-Lik), Akaike information criterion (AIC), Kolmogorov-Smirnov (K-S) test Statistic, Anderson Darling (A^*) test Statistic, Crammer von mises (W^*) test Statistic and Probability-Probability (P-P) plots were used as criteria for comparison.

Keywords: Lindley distribution, Beta-*G*, Exponentiated-*G*, Kumaraswamy-*G*, Primary 60E05

INTRODUCTION

Lifetime data analysis is a statistical method for analyzing real lifetime problem based on survival time. This survival time deals with the time to the occurrence of a given event; which can be the development of a disease, the response to treatment or failure time of a component in an engine. In recent decades, the study of survival data is centered on predicting the probability of response, survival or mean lifetime of certain characteristic of interest. These predictions are attainable through the use of lifetime distributions. Generally, lifetime distributions seek to analyze time-to-event data. Many lifetime distributions have been introduced, studied and applied in literature to model lifetime data and examples of such include the exponential, beta, gamma, Gumbel, Weibull distribution, etc. In this paper, the authors wish to explore some generalized Lindley distributions arising from different methods of generalizing classical lifetime distribution. Lindley (1958) proposed the classical one parameter Lindley distribution with probability density function define as

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \qquad x>0,\ \theta>0 \qquad (1)$$

and the cumulative distribution function given by

$$F(x) = 1-\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}, \qquad x>0,\ \theta>0 \tag{2}$$

The density function defined in Equation (1) can be expressed as a two-component mixture of exponential (θ) and gamma (2, θ) distribution in the form

$$f(x) = pf_1(x) + (1-p)f_2(x) \tag{3}$$

where $f_1(x)$ and $f_2(x)$ are the density functions of the exponential (θ) and Gamma (2, θ) distribution respectively, p is the mixing proportion (normalizing constant). Figure 1 shows the pdf of the Lindley distribution for selected value of the parameter θ.

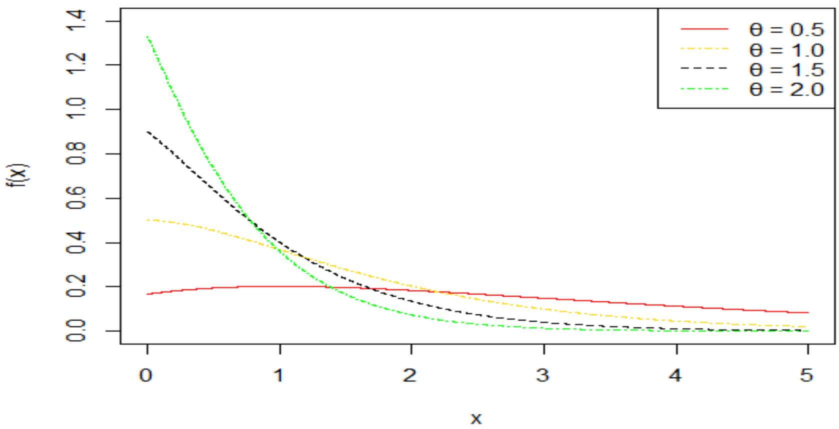

Figure 1: Density plots of the Lindley distribution for different values of parameter θ

Figure 1 clearly shows that the plot of the density function can be monotonically decreasing (reversed J-shape). Given the pdf and cdf in Equations (1) and (2), the hazard rate function of the Lindley distribution is defined as

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\theta^2(1+x)}{\theta+1+\theta x}$$

(4)

The plot of the hazard rate function of the Lindley distribution is displayed in Figure 2.

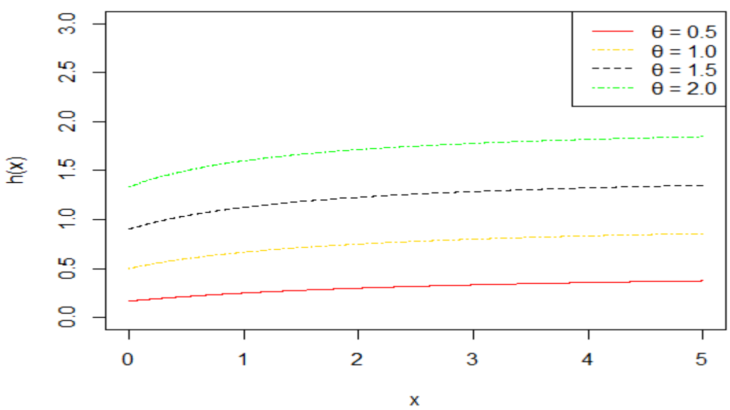

Figure 2: Hazard rate function of the Lindley distribution for different values of parameter θ

The plot shows that, at different choice of the parameter, the shape of the hazard rate function of the Lindley distribution is strictly monotonically increasing. Ghitany *et al.* (2008) gave a comprehensive treatment of the mathematical properties of the Lindley distribution and showed in many ways that the distribution is a better model than the one based on the exponential distribution. The remaining Sections of this paper are organized as follows: Section 2 presents a review of some well-known Lindley family of distributions generated by using some methods of generalizing classical lifetime distributions. In Section 3, the authors considered the applicability of the generalized Lindley distributions in real life data fitting. Finally, Section 4 concludes the work.

Methods of Generating New Statistical Distribution

The Mixture Method

This method is based on mixture of two probability density functions with a mixing proportion, as given in Equation (3). Table 1 shows some generalized distributions derived using this method.

Table 1: Distributions arising from the mixture method

Distribution	Density function (pdf)	Mixing pdfs	Authors
New generalized Lindley distribution (NGLD)	$\frac{1}{(\theta+1)}\left(\frac{\theta^{\alpha+1}x^{\alpha-1}}{\Gamma(\alpha)}+\frac{\theta^{\beta}x^{\beta-1}}{\Gamma(\beta)}\right)e^{-\theta x}$	Gamma (θ, α) & Gamma (θ, β)	Elbatal *et al.* (2013)
Quasi Lindley distribution (QLD)	$\frac{\theta(\alpha+x\theta)}{\alpha+1}e^{-\theta x}$	Gamma (2, θ) & Exponential (θ)	Shanker and Mishra (2013)
Power Lindley distribution (PLD)	$\frac{\alpha\beta^2}{\beta+1}(1+x^{\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}$	Gen. Gamma(2, β, α) & Weibull (α, β)	Ghitany et al., (2013)
New two-parameter generalized Lindley distribution (NTPGLD)	$\frac{\theta^2}{(\theta+1)}\left(1+\frac{\theta^{\alpha-2}x^{\alpha-1}}{\Gamma(\alpha)}\right)e^{-\theta x}$	Gamma(θ, α) & Exponential (θ)	Ekhosuehi *et al.* (2018)

Each of these generalized Lindley distribution aims at increasing the flexibility of the Lindley distribution. Although, an application of the Quasi Lindley distribution suggests that the distribution provides better fit than the classical Lindley distribution, the plots of its density function maintain a right-skewed unimodal shape and a strictly increasing failure rate functions as in the case of the Lindley distribution.

The Kumaraswamy-G Method

Kumaraswamy (1980) introduced the Kumaraswamy distribution defined on a unit interval (0,1) with the cdf defined as

$$G(x) = 1-\left(1-x^{\alpha}\right)^{\beta}, \qquad x\in(0,1) \tag{5}$$

and the corresponding pdf defined as

$$g(x) \quad = \quad \alpha\,\beta\,x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta-1} \ , \qquad \alpha,\,\beta>0,\ x>0. \tag{6}$$

Suppose $G(x)$ denotes the baseline cumulative function of a random variable, Cordeiro and de Castro (2011) introduced a generalized form of the distribution called Kumaraswamy-*G* distribution with cdf defined as

$$F(x) \quad = \quad 1-\left(1-[G(x)]^{\alpha}\right)^{\beta}, \tag{7}$$

and pdf given by

$$f(x) \quad = \quad \alpha\,\beta\,x^{\alpha-1}\left(1-[G(x)]^{\alpha}\right)^{\beta-1}[G(x)]^{\alpha-1}g(x)\ , \qquad \alpha,\,\beta>0,\ x>0. \tag{8}$$

Using the technique in Equations (7) and (8), Table 2 gives some generalizations of the Lindley distribution.

Table 2: Distributions arising from the Kumaraswamy-*G* Method

Distributions	Cummlative distribution function	Authors
Kumaraswamy Quasi Lindley distribution (KQLD)	$1-\left\{1-\left(1-\frac{(1+\lambda+\theta\,x)}{\lambda+1}e^{-\theta\,x}\right)^{\alpha}\right\}^{\beta}$	Elbatal and Elgarhy (2013)
Kumaraswamy power Lindley Distribution (KPLD)	$1-\left\{1-\left(1-\frac{(1+\theta+\theta\,x^{\lambda})}{\theta+1}e^{-\theta\,x^{\lambda}}\right)^{\alpha}\right\}^{\beta}$	Oluyede *et al.* (2016)
Kumaraswamy Lindley distribution (KLD)	$1-\left\{1-\left(1-\frac{(1+\theta+\theta\,x)}{\theta+1}e^{-\theta\,x}\right)^{\alpha}\right\}^{\beta}$	Salem and Hagag (2017)
Kumaraswamy Unit-Gompertz Distribution (KUGD)	$1-\left[1-e^{-\gamma(x^{-\beta}-1)}\right]^{\lambda}$	Opone et al. (2023)

Motivations for using this method of generalization arose from the work of Jones (2009), who gave a comprehensive background of the Kumaraswamy distribution, and more importantly, pointed out some advantages of the Kumaraswamy distribution over the beta distribution. Although, the two distributions are defined on a unit interval (0,1), the Kumaraswamy distribution has an explicit expression for the cumulative distribution function and the quantile function which does not involve special functions.

The Exponentiated-G Method

The exponentiated class of distributions was first reported in the work of Mudholkar and Srivastava (1993) who proposed the exponentiated Weibull family for analyzing Bathtub failure rate data. The cumulative distribution function is defined by

$$F(x)=\left[G(x)\right]^{\alpha} \tag{9}$$

Cordeiro *et al.* (2013) introduced a variant of the class of distributions with cdf given by

$$F(x) = \left\{1-\left[1-G(x)\right]^{\alpha}\right\}^{\beta}, \qquad \alpha, \beta > 0, \; x > 0, \tag{10}$$

and probability density function

$$f(x) = \alpha\beta\left[1-G(x)\right]^{\alpha-1}\left\{1-\left[1-G(x)\right]^{\alpha}\right\}^{\beta-1} g(x), \quad \alpha, \beta > 0, \; x > 0. \tag{11}$$

Table 3 gives some generalized Lindley distribution arising from this method.

Table 3: Distributions arising from the Exponentiated-G method

Distributions	Cumulative distribution function	Authors
Generalized Lindley distribution (GLD)	$\left\{1-\frac{(1+\theta+\theta x)}{\theta+1}e^{-\theta x}\right\}^{\beta}$	Nadarajah *et al.* (2011)
Exponentiated power Lindley distribution (EPLD)	$\left\{1-\frac{(1+\theta+\theta x^{\alpha})}{\theta+1}e^{-\theta x^{\alpha}}\right\}^{\beta}$	Warahena-Liyanage and Pararai (2014)
Exponentiated Quasi Lindley distribution (EQLD)	$\left\{1-\frac{(1+\alpha+\theta x)}{\alpha+1}e^{-\theta x}\right\}^{\beta}$	Elbatal *et al.* (2016)

The exponentiated class of distributions is known to have a unique property of accommodating both monotone and non-monotone hazard rate functions. When the exponentiated (shape) parameter is less than 1, the distribution exhibits a decreasing hazard rate property and increasing hazard rate property when the shape parameter is greater 1. This class of distributions also demonstrates flexibility in handling a right skewed data set.

The Odd Log-Logistic-G Method

Gleaton and Lynch (2006) introduced a new class of distributions called "the odd log-logistic family of distribution". The cumulative distribution function of the family is given by

$$F(x,\alpha,\xi) = \frac{G(x,\xi)^{\alpha}}{G(x,\xi)^{\alpha} + \overline{G}(x,\xi)^{\alpha}} \tag{12}$$

where $\overline{G}(x)$ is the survival function of the baseline distribution and α, ξ are parameter vectors.

Gleaton and Lynch (2010) further gave a modification of the OLL-G by introducing an extra parameter and called it the odd log-logistic Marshall-Olkin (OLLMO) family of distributions. The cdf of the OLLMO family of distributions is given by

$$F(x,\alpha,\xi) = \frac{G(x,\xi)^{\alpha}}{G(x,\xi)^{\alpha} + \beta\overline{G}(x,\xi)^{\alpha}} \tag{13}$$

Some generalized Lindley distributions developed using this method is shown in Table 4.

Table 4: Distributions Arising from the Odd Log-Logistic-*G* Method

Distributions	Cumulative distribution function	Authors
OLL- Lindley distribution (OLL-LD)	$\dfrac{\left\{1-\frac{(1+\theta+\theta x)}{\theta+1}e^{-\theta x}\right\}^{\alpha}}{\left\{1-\frac{(1+\theta+\theta x)}{\theta+1}e^{-\theta x}\right\}^{\alpha}+\left\{\frac{(1+\theta+\theta x)}{\theta+1}e^{-\theta x}\right\}^{\alpha}}$	Ozel *et al.* (2017)
OLLP-Lindley distr. (OLLPLD)	$\dfrac{\left\{1-\frac{(1+\theta+\theta x^{\lambda})}{\theta+1}e^{-\theta x^{\lambda}}\right\}^{\alpha}}{\left\{1-\frac{(1+\theta+\theta x^{\lambda})}{\theta+1}e^{-\theta x^{\lambda}}\right\}^{\alpha}+\left\{\frac{(1+\theta+\theta x^{\lambda})}{\theta+1}e^{-\theta x^{\lambda}}\right\}^{\alpha}}$	Alizadeh *et al.* (2017b)
OLL-MO power Lindley Distr.	$\dfrac{\left\{1-\frac{(1+\theta+\theta x^{\lambda})}{\theta+1}e^{-\theta x^{\lambda}}\right\}^{\alpha}}{\left\{1-\frac{(1+\theta+\theta x^{\lambda})}{\theta+1}e^{-\theta x^{\lambda}}\right\}^{\alpha}+\beta\left\{\frac{(1+\theta+\theta x^{\lambda})}{\theta+1}e^{-\theta x^{\lambda}}\right\}^{\alpha}}$	Alizadeh *et al.* (2017a)

Figure 3 shows that this class of distributions span the various shapes of the failure rate property, which can be decreasing, increasing, bathtub and upside-down bathtub (unimodal) shapes.

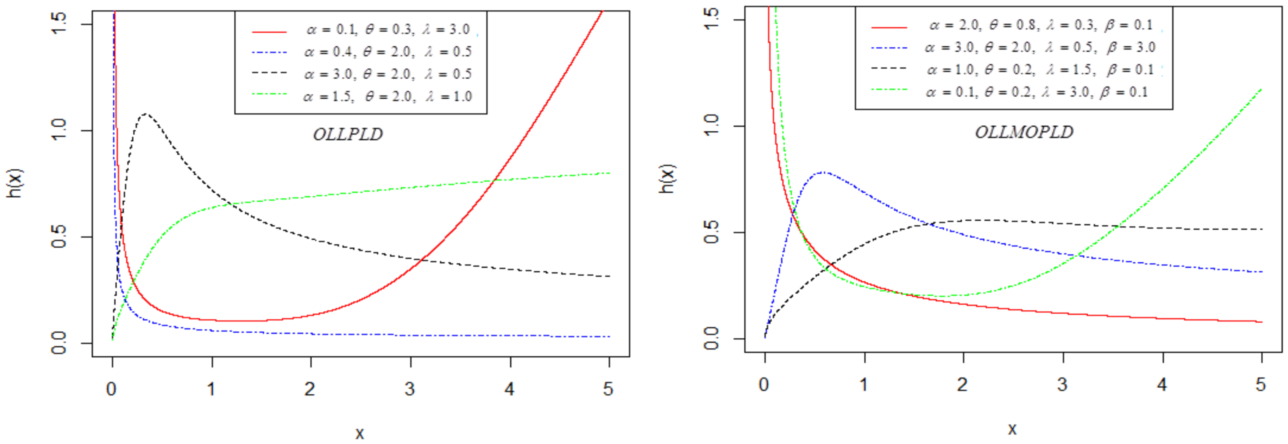

Figure 3: The hazard rate function of the OLPLLD and OLLMOPLD for different values of parameter

The Beta-G Method

Let $F(x)$ be the cumulative distribution function of a random variable X and $r(t)$ be the density function of a random variable *T*. Eugene *et al.* (2002) defined the cdf of a generalized class of distributions as

$$G(x) = \int_0^{F(x)} r(t)\, dt\,, \qquad 0<t<1 \qquad (14)$$

In particular,

$$G(x) = \frac{1}{B(a,b)} \int_0^{F(x)} t^{a-1}(1-t)^{b-1}\, dt \tag{15}$$

and the corresponding density function is given by

$$g(x) = \frac{f(x)}{B(a,b)} \left[F(x)\right]^{a-1} \left[1-G(x)\right]^{b-1} \tag{16}$$

Equations (15) and (16) are readily the cdf and pdf of the Beta-*G* class of distribution. Table 5 shows some generalized Lindley distribution arising from the Beta-*G* class of distribution.

Table 5: Distributions arising from the Beta-*G* Method

Distributions	Probability Density Function (pdf)	Authors
New generalized Lindley distribution (NGLD)	$\frac{\theta^2}{B(a,b)(\theta+1)}(1+x)e^{-\theta x}\left[G(x)\right]^{a-1}\left[G(x)\right]^{b-1}$	MirMostafaee *et al.* (2015)
Beta exponentiated Lindley distribution (BELD)	$\frac{\theta^2 \lambda}{B(a,b)(\theta+1)}(1+x)e^{-\theta x}\left[G(x)\right]^{\lambda a-1}\left[1-[G(x)]^{\lambda}\right]^{b-1}$	Rodrigues *et al.* (2015)
Beta power Lindley distribution (BPLD)	$\frac{\alpha\theta^2}{B(a,b)(\theta+1)}\left(1+x^{\alpha}\right)x^{\alpha-1}e^{-\theta x^{\alpha}}\left[G(x)\right]^{a-1}\left[1-G(x)\right]^{b-1}$	Pararai *et al.* (2015b)

Jones (2009) made clear some similarities and differences between the Kumaraswamy distribution and beta distribution and highlighted some advantages of the Kumaraswamy distribution over the beta distribution. Notwithstanding, the beta distribution has the following advantages over the Kumaraswamy distribution: simpler expression for the moments and moment generating function, a one-parameter sub-family of symmetric distribution and a simpler moment estimation. Generally, the density function of the beta distribution accommodates a right skewed, left skewed and a symmetric unimodal shape as reported in Opone and Ekhosuehi (2017).

The Transformed-Transformer (*T-X*) Method

An extension of the Beta-G class of distribution is the one based on the Transformed-Transformer (*T-X*) family of distributions proposed by Alzaatreh *et al.* (2013). The cumulative distribution function of the *T-X* family of distributions is defined by

$$G(x) = \int_a^{W[F(x)]} r(t)dt, \qquad -\infty < t < \infty \tag{17}$$

and the corresponding density function as

$$g(x) = \frac{d}{dx} W[F(x)] r\{W[F(x)]\} \tag{19}$$

where $W[F(x)]$ is a differentiable and monotonically non-decreasing function.

Table 6: Distributions arising from the *T-X* Method

Distributions	Probability Density Function (pdf)	Authors
Lindley-Pareto distribution (LPD)	$\dfrac{\lambda\theta^2 e^{\theta} x^{2\lambda-1} e^{-\theta\left(\frac{x}{\alpha}\right)\lambda}}{\alpha^{2\lambda}(\theta+1)}$	Lazri and Zeghdoudi (2016)
odd Lindley Burr XII distribution (OLBXII)	$\dfrac{\alpha\beta\theta^2 e^{\theta} x^{\alpha-1}(1+x^{\alpha})^{2\beta-1} e^{-\theta(1+x^{\alpha})\beta}}{(\theta+1)}$	Korkmaz *et al.* (2018)
Three-parameter generalized Lindley distribution (TPGLD)	$\dfrac{\alpha\theta^2}{(\theta\beta+1)}(\beta+x^{\alpha})x^{\alpha-1}e^{-\theta x^{\alpha}}$	Ekhosuehi and Opone (2018)

The *T-X* method of generalization has remained the widest method used not only in generalizing the Lindley distribution but also other classical statistical distributions. The density function of the generalized distributions using this method has a unique property of exhibiting a left-skewed, right-skewed, symmetric and reversed J shapes as displayed in Figure 4.

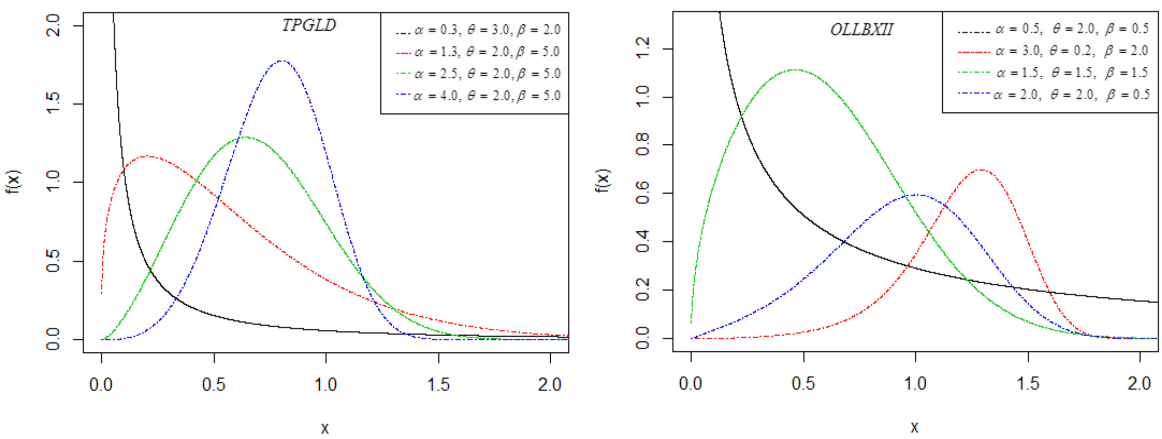

Figure 4: Density function of the TPGLD and OLLBXII for different values of parameter

Data Analysis

In this Section, the authors considered an application of some generalized Lindley distributions using a remission time data set. These distributions include; Lindley distribution due to Lindley (1958), generalized Lindley distribution (GLD), power Lindley distribution (PLD), exponentiated power Lindley distribution (EXPLD), Kumaraswamy power Lindley distribution (KPLD), Kumaraswamy Lindley distribution (KLD), odd log-logistic Marshall-Olkin power Lindley distribution (OLLMOPLD), odd log-logistic power Lindley distribution (OLLPLD) and odd Lindley Bur XII distribution (OLBXII). The fit of the distributions for the data set are compared using the maximized log-likelihood (*Log-Lik*), Akaike Information Criterion (*AIC*), Kolmogorov-Smirnov (*K-S*) test statistic, Crammer-von-Mises (W^*) test statistic, Anderson Darling (*A**) test statistic and Probability-Probability (*p-p*) plots.

Data Set: Bladder Cancer Patient

The data set in Table 7 represents an uncensored data set corresponding to the remission times (in months) of 128 bladder cancer patients. The data set was first reported in Lee and Wang (2003). Alizadeh *et al.* (2017b) also used the data set to show the applicability of the odd log-logistic power Lindley distribution.

Table 7: Bladder Cancer Patients Data

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98	6.97	9.02
13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.8	25.74	0.50	2.46	3.64	5.09	7.26
9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81	2.62	3.82
5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69
4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33	5.49	7.66	11.25	17.14
79.05	1.35	2.87	5.62	7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93	11.79	18.1
1.46	4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02	2.02	3.31
4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07	21.73	2.07	3.36	6.93	8.65
12.63	22.69	-	-	-	-	-	-	-	-	-	-	-	-

Table 8: Comparison Criteria for the Bladder Cancer Patient Data

Models	*Estimates*	*Log-Lik*	*AIC*	*K-S* *(p-value)*	W^* *(p-value)*	A^* *(p-value)*
LINDLEY	$\beta = 0.1960$	-419.5429	841.0858	0.1165 (0.0621)	0.5195 (0.0354)	2.7871 (0.0352)
PLD	$\alpha = 0.8303$ $\beta = 0.2942$	-419.3662	830.7325	0.0684 (0.5877)	0.1270 (0.4683)	0.7887 (0.4890)
GLD	$\alpha = 0.7334$ $\beta = 0.1648$	-416.2975	836.5950	0.0928 (0.2202)	0.2467 (0.1927)	1.3232 (0.2249)
KLD	$\beta = 0.5015$ $\lambda = 0.9755$ $\theta = 0.2832$	-414.1156	834.2312	0.0907 (0.2426)	0.2487 (0.1901)	1.2369 (0.2538)
OLLPLD	$\alpha = 0.4064$ $\beta = 0.6054$ $\lambda = 2.5346$	-409.4500	824.9000	0.0316 (0.9995)	0.0147 (0.9997)	0.0970 (1.0000)
EXPLD	$\alpha = 0.5665$ $\beta = 0.8184$ $\lambda = 2.7652$	-410.4480	826.8959	0.0429 (0.9724)	0.0362 (0.9522)	0.2401 (0.9754)
OLBXII	$\alpha = 1.5460$ $\beta = 0.5046$ $\lambda = 0.4406$	-411.0979	828.1957	0.0507 (0.8971)	0.0575 (0.8307)	0.3666 (0.8809)

OLLMOPLD	$\alpha = 0.3856$ $\beta = 0.6661$ $\lambda = 2.6242$ $\theta = 1.4034$	-409.4491	826.8983	0.0316 (0.9995)	0.0147 (0.9997)	0.0967 (1.0000)
KPLD	$\alpha = 0.5079$ $\beta = 0.8217$ $\lambda = 3.1126$ $\theta = 1.4212$	-410.4297	828.8595	0.0430 (0.9720)	0.0356 (0.9549)	0.2367 (0.977)

The fits of the Probability-Probability (p-p) plots of each distribution for the bladder cancer patient data is given in the Figure 5.

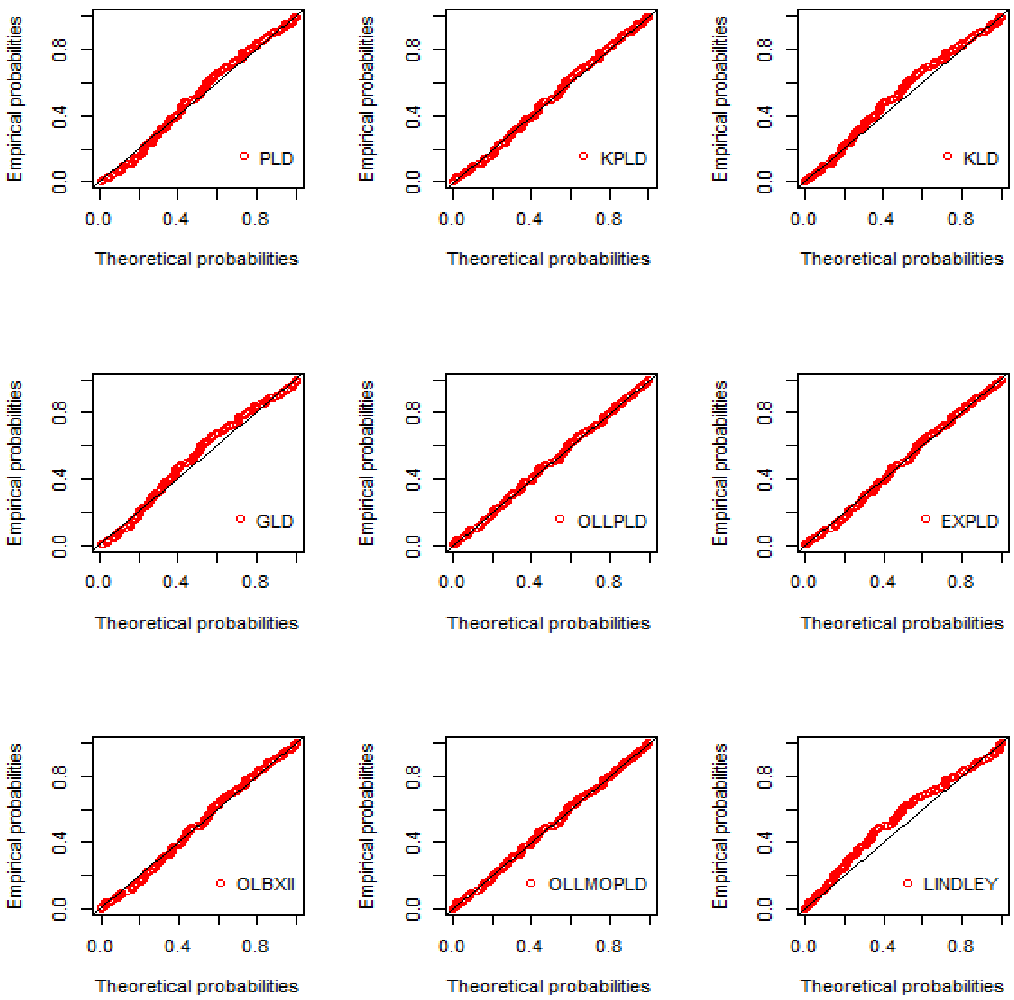

Figure 5: The Probability-Probability (p-p) plots of the distributions for the bladder cancer patient data

Discussion of Results

A suitable model for analyzing lifetime data can be investigated among several distributions by examining the model with the maximized log-likelihood value and the least Akaike Information Criterion (*AIC*), Kolmogorov-Smirnov test statistic (*K-S*), Anderson Darling (A^*) test statistic and Crammer von Mises (W^*) test statistic. Table 8 reveals that the odd log-logistic power Lindley distribution (OLLPLD) outperformed the rest distributions in the bladder cancer patient data set. This claim was further supported by examining the Probability-Probability (*P-P*) plots of the distributions for the data sets as displayed in Figure 6.

Conclusion

In this paper, the authors present a review on some methods of developing new generalization of statistical distributions, in particular, the Lindley family of distributions. Some well-known generalized Lindley distributions were established and an application of some of the distributions to a real data set reveals that the odd log-logistic power Lindley distribution (OLLPLD) provides better fit than all the nested distributions under study in fitting the bladder cancer data set.

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